Thus,

$$\tilde{N}_f = \begin{bmatrix} -0.6E(z_1^3) - 0.8E(z_1^2)\bar{z}_2 + \bar{z}_3\bar{z}_1 - 0.6E(z_1^2) + 0.8\bar{z}_1\bar{z}_2 - \bar{z}_3 \\ -0.36E(z_1^3) + 0.6E(z_1^2)0.96E(z_1^2)\bar{z}_2 + 0.8\bar{z}_1\bar{z}_2 - \bar{z}_3 + 0.6\bar{z}_1\bar{z}_3 - 0.64\bar{z}_1E(z_1^2) + 0.8\bar{z}_2\bar{z}_3 \\ -0.6E(z_1^2)\bar{z}_3 - 0.8\bar{z}_1\bar{z}_2\bar{z}_3 + E(z_3^2) + 0.6E(z_1^2) + 0.8\bar{z}_1\bar{z}_2 - \bar{z}_3 \end{bmatrix}^T P^{-1}$$

$$= [-1.0 \quad -1.0 \quad 1.0]$$

Equation (28) can again be reduced to a simple integral by first integrating with respect to dz_1 as in Eq. (27).

C. $f(\xi) = L(\xi_1, \ldots, \xi_{n-1})g(\xi_n)$, Where $\xi = [\xi_1^T \ldots \xi_n^T]^T$, L Is a Multivariable Polynomial in $\xi_1, \ldots, \xi_{n-1}$ and g Is a Scalar Nonlinear Function

Here we decompose P as $P = RR^T$ with R upper triangular. Note that such decomposition can be obtained by performing the Cholesky decomposition on P^{-1}.[7] Let $z = R^{-1}\xi$ and the remaining manipulations are the same as in Secs. IVA and IVB.

V. Illustrative Example

Let us consider the three-input nonlinearity described by

$$F(x) = -x_1x_2 + x_3 \tag{29}$$

where $x = [x_1\ x_2\ x_3]^T$. Note that Eq. (29) has the same form as the right side of the moment Eqs. (23–25). Assume that the random vector x is jointly Gaussian with mean $\bar{x} := Ex = [1\ \ 1\ \ 1]^T$ and covariance matrix

$$P := E(x-\bar{x})(x-\bar{x})^T = \begin{bmatrix} 1 & 0.6 & 0 \\ 0.6 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

We wish to find $\bar{N}_f \in \mathbb{R}^{1\times 3}$ and $\tilde{N}_f \in \mathbb{R}^{1\times 3}$ such that the linear function $L(x) = \bar{N}_f\bar{x} + \tilde{N}_f(x-\bar{x})$ is the best linear approximation to $f(x)$.

To compute $\bar{N}_f$ and $\tilde{N}_f$ by Eq. (16) and Eqs. (18–21), first compute the Cholesky factor R of P to get

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and let $z := R^{-1}x$, $\bar{z} := R^{-1}\bar{x}$. The random variable $z = [z_1\ z_2\ z_3]^T$ is jointly Gaussian with covariance matrix equal to identity and $\bar{z} = Ez = [1\ \ 0.5\ \ 1]$. Let $f(z) := F(Rz) = -0.6z_1^2 - 0.8z_1z_2 + z_3$. We have

$$EF(z) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (0.6z_1^2 - 0.8z_1z_2 + z_3)$$
$$\times \exp[-½\textstyle\sum_{k=1}^{3}(z_k-\bar{z}_k)^2]\ dz_1\ dz_2\ dz_3 \tag{30}$$
$$= 0.6E(z_1^2) - 0.8\bar{z}_1\bar{z}_2 + \bar{z}_3 = 1.8$$

where

$$E(z_1^2) = \frac{1}{(2\pi)^{½}} \int_{-\infty}^{\infty} z_1^2 \exp[-(z_1-\bar{z}_1)^2]\ dz_1$$

Note that the multiple integral in Eq. (30) can be decomposed as a sum of products of simple integrals. Thus, by Eqs. (16) and (20),

$$\bar{N}_f = (\bar{x}^T\bar{x})^{-1}\bar{x}^T Ef(x) = (\bar{x}^T\bar{x})^{-1}\bar{x}^T EF(z)$$
$$= [-0.2 \quad -0.2 \quad -0.2]$$

To compute $\tilde{N}_f$ by Eq. (21), expand

$$[R(z-\bar{z})]^TF(z) = [z_1 - 1 \quad 0.6z_1 + 0.8z_2 - 1 \quad z_3 - 1]$$
$$\times (-0.6z_1^2 - 0.8z_1z_2 + z_3)$$

Note that the high-order moments of the random variables are obtained by using the characteristic function.[9] This gives the optimal equivalent gain for fixed mean and covariance matrix. If such a computation is carried out for a few different values of mean and covariance, the curve-fitting method can then be used to obtain a closed-form approximation to the random-input describing function for the multi-input nonlinearity.

VI. Summary

In this Note, we give a straightforward yet rigorous derivation of the formulas for computing the optimal linear equivalent gains for multi-input/multi-output nonlinearities with random inputs. For general nonlinearities, the computation requires the evaluation of multiple integrals. It is shown that, for some classes of nonlinearities with Gaussian signal inputs, the computation can be greatly simplified and readily carried out numerically. This result has been used in the covariance analysis of interconnected nonlinear systems with multi-input nonlinearities.

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Constant Covariance in Local Vertical Coordinates for Near-Circular Orbits

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Introduction

MANY covariance studies involve near-circular orbits, and it is often desirable to be able to initialize an error covariance matrix that, in the absence of measurements, will remain constant in a rotating local-vertical coordinate system. This Note describes a way to define such a covariance matrix

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equation simplicity, solution visibility (e.g., up is always up), and it also preserves the desirable derivative relationship between the position and velocity terms. When using these equations purely for extrapolation, increased accuracy can be obtained by defining the state deviations in terms of a curvilinear (or cylindrical) coordinate system with x along the nominal circular orbit, and y radially perpendicular to the trajectory.

Clearly, the coefficient matrices in Eqs. (4) and (5) represent the state transition matrix, in rotating coordinates, for circular orbits. The state transition matrix can be written in a full six-dimensional format as follows:

$$\Phi \equiv \begin{bmatrix} 1 & 6(\sin\omega t - \omega t) & 0 & (1/\omega)(4\sin\omega t - 3\omega t) & (2/\omega)(\cos\omega t - 1) & 0 \\ 0 & 4 - 3\cos\omega t & 0 & (2/\omega)(1 - \cos\omega t) & (1/\omega)\sin\omega t & 0 \\ 0 & 0 & \cos\omega t & 0 & 0 & (1/\omega)\sin\omega t \\ 0 & 6\omega(\cos\omega t - 1) & 0 & 4\cos\omega t - 3 & -2\sin\omega t & 0 \\ 0 & 3\omega\sin\omega t & 0 & 2\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & -\omega\sin\omega t & 0 & 0 & \cos\omega t \end{bmatrix} \tag{6}$$

including the proper cross-correlation terms. The method includes the possibility of having small period errors that cause the error ellipsoid (as represented by the covariance matrix) to grow, but otherwise rotate with the local-vertical frame. Additionally, for convenience, the principal results of this Note are reformulated in a nonrotating local-vertical coordinate system in the Appendix.

Clohessy-Wiltshire Equations

The differential equations describing the motion of small state deviations (in a rotating local-vertical coordinate system) away from a nominally circular orbit are known as the Clohessy-Wiltshire (CW) equations[1] (see Kaplan[2] for a derivation):

$$\ddot{x} + 2\omega\dot{y} = 0 \tag{1}$$

$$\ddot{y} - 2\omega\dot{x} - 3\omega^2 y = 0 \tag{2}$$

$$\ddot{z} + \omega^2 z = 0 \tag{3}$$

Here, x, y, and z are position deviations in a local-vertical coordinate system that is rotating at orbital rate ω as shown in Fig. 1. It should be noted that, because of a different coordinate system definition, there are a few sign differences between these equations and those given in the reference. The familiar dot notation represents the usual differentiation with respect to time, and the solution to these equations is easily verified to be

$$\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 6(\sin\omega t - \omega t) & (1/\omega)(4\sin\omega t - 3\omega t) & (2/\omega)(\cos\omega t - 1) \\ 0 & 4 - 3\cos\omega t & (2/\omega)(1 - \cos\omega t) & (1/\omega)\sin\omega t \\ 0 & 6\omega(\cos\omega t - 1) & 4\cos\omega t - 3 & -2\sin\omega t \\ 0 & 3\omega\sin\omega t & 2\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ \dot{x}_0 \\ \dot{y}_0 \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} z \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \cos\omega t & (1/\omega)\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} z_0 \\ \dot{z}_0 \end{bmatrix} \tag{5}$$

These matrix equations represent state deviation extrapolation in a rotating local-vertical coordinate system, and have been partitioned in this way to emphasize that the out-of-plane motion is completely decoupled from the in-plane motion. The use of a rotating frame offers advantages in terms of

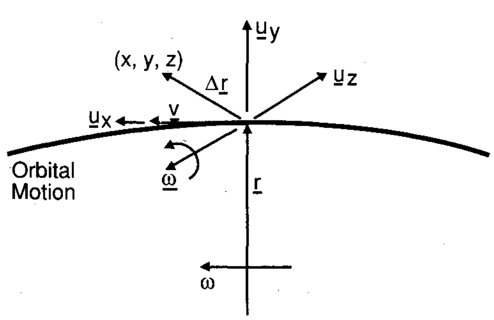

Fig. 1 Local-vertical rotating coordinate system.

with the six-dimensional state deviations extrapolated by means of

$$\Delta\underline{x} = \Phi\Delta\underline{x}_0 \tag{7}$$

From this, it follows that the state transition matrix may also be used to extrapolate the error covariance matrix (E) by means of the familiar relationship

$$E \equiv \overline{\Delta\underline{x}\Delta\underline{x}^T} = \Phi E_0 \Phi^T \tag{8}$$

The purpose of this Note is to find E such that it remains symbolically invariant under this extrapolation. This is another way of saying that the form of E should be constant in the local-vertical frame.

Traveling Ellipse Formulation

With some algebraic manipulation, the solution to the CW equations may be rewritten in a very compact form[3,4] that will prove essential to what follows:

$$x = X_0 - \frac{3}{2}\omega t Y_0 + 2b\cos(\omega t + \phi) \tag{9}$$

$$y = Y_0 + b\sin(\omega t + \phi) \tag{10}$$

$$z = c\sin(\omega t + \psi) \tag{11}$$

$$\dot{x} = -\frac{3}{2}\omega Y_0 - 2\omega b\sin(\omega t + \phi) \tag{12}$$

$$\dot{y} = \omega b\cos(\omega t + \phi) \tag{13}$$

$$\dot{z} = \omega c\cos(\omega t + \psi) \tag{14}$$

The out-of-plane motion can be described as a simple oscillator, and the in-plane motion as an ellipse which may be translating horizontally. In fact, the in-plane equations imply that the state deviations trace out a 2×1 ellipse, with b as the semiminor-axis, which moves horizontally if the parameter Y_0 is nonzero. For this reason, these equations are sometimes known as the traveling ellipse form of the Clohessy-Wiltshire equations (see Figs. 2 and 3 for a graphical depiction of the in-plane motion). Note that the ellipse travels to the right (falls behind) if higher than nominal, and to the left (moves ahead) if lower than nominal. Essentially, Y_0 represents the deviation in orbital semimajor-axis (r), and the ellipse moves forward or backward depending on the sign of this term. Thus, it should

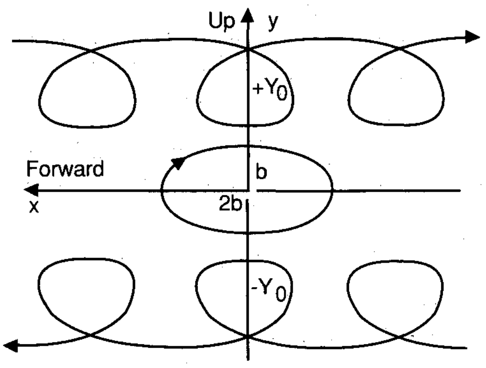

Fig. 2 In-plane traveling ellipse—vertical deviations.

not be too surprising that Y_0 is closely related to the period deviation ΔP. In fact, it is easily shown that

$$\Delta P = \frac{3P}{2r} Y_0 \tag{15}$$

In contrast, X_0 determines the location of the 2×1 ellipse along the trajectory and, as such, represents a purely downrange deviation that has no effect on the orbital motion. The parameter c is the magnitude of the simple oscillator out-of-plane motion, and the parameters ϕ and ψ are the phase angles for the in-plane and out-of-plane motions, respectively.

These new parameters may be obtained from the initial state deviations by means of the following formulas:

$$X_0 = x_0 - (2/\omega)\dot{y}_0 \tag{16}$$

$$Y_0 = 4y_0 + (2/\omega)\dot{x}_0 \tag{17}$$

with b, c, ϕ, and ψ determined from

$$b \cos\phi = (1/\omega)\dot{y}_0 \tag{18}$$

$$b \sin\phi = -3y_0 - (2/\omega)\dot{x}_0 \tag{19}$$

$$c \cos\psi = z_0 \tag{20}$$

$$c \sin\psi = (1/\omega)\dot{z}_0 \tag{21}$$

The point here is that the complete state deviation may be described equally well by either parameter set, that is (x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$) or (b, c, X_0, Y_0, ϕ, ψ). However, the latter parameter set has some very nice geometric interpretations that will prove very useful in the next section.

Covariance Interpretation

In order to design an error covariance matrix, it is necessary to understand that the state deviation parameters (x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$), reinterpreted as errors, are not all independent. However, it is not at all clear what the correlations between these parameters should be. But, with the new geometric parameter set (b, c, X_0, Y_0, ϕ, ψ), it is possible to make some intelligent *assumptions* about what statistical dependence or independence among the parameters is needed in order to design an error covariance matrix with certain desired characteristics.

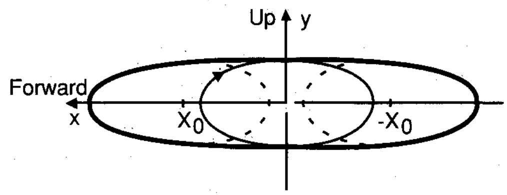

Fig. 3 In-plane no period error—downrange deviations.

The quantities b and c, and X_0 and Y_0 are parameter pairs that describe the size and center, respectively, of the relative motion. However, the parameters ϕ and ψ are phase angles describing where the actual state is located. Without a priori knowledge, these phase angles must be assumed to be completely independent and uniformly distributed random variables. Thus, with the aid of Eqs. (9–14), the individual elements of an error covariance matrix may be obtained from the expression

$$E_{ij} = \overline{\Delta x_i \Delta x_j} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \Delta x_i \Delta x_j \, d\phi \, d\psi \qquad 1 \le i, j \le 6 \tag{22}$$

which usually reduces to a single integral. In any case, by defining

$$X_t \equiv X_0 - \frac{3}{2}\omega t Y_0 \tag{23}$$

it may be shown that

$$E_t = \begin{bmatrix} \overline{X_t^2} + 2b^2 & \overline{X_t Y_0} & 0 & -(3\omega/2)\overline{X_t Y_0} & \omega b^2 & 0 \\ \overline{X_t Y_0} & Y_0^2 + (1/2)b^2 & 0 & -\omega[(3/2)Y_0^2 + b^2] & 0 & 0 \\ 0 & 0 & (1/2)c^2 & 0 & 0 & 0 \\ -(3\omega/2)\overline{X_t Y_0} & -\omega[(3/2)Y_0^2 + b^2] & 0 & \omega^2[(9/4)Y_0^2 + 2b^2] & 0 & 0 \\ \omega b^2 & 0 & 0 & 0 & (\omega^2/2)b^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\omega^2/2)c^2 \end{bmatrix} \tag{24}$$

The integration, or averaging, over the phase angles eliminates all of the sinusoidal terms from the problem. However, expectation bars have been left over any "mixed" terms that contain potential cross correlations. In particular, the $X_0 Y_0$ terms require additional consideration. If k is defined as the correlation coefficient between these two parameters, then it follows that

$$\overline{X_0 Y_0} = k \sqrt{\overline{X_0^2}\,\overline{Y_0^2}} = k X_0 Y_0 \tag{25}$$

Note the somewhat loose notation used here. The bars have been dropped over the quantities X_0 and Y_0 which are now to be interpreted as standard deviations of some statistical parameters. This is also true of the parameters b and c appearing in Eq. (24). Now, using Eqs. (23) and (25), it follows that

$$\overline{X_t Y_0} = k X_0 Y_0 - \frac{3}{2}\omega t Y_0^2 \tag{26}$$

$$\overline{X_t X_t} = X_0^2 - 3\omega t k X_0 Y_0 + \frac{9}{4}\omega^2 t^2 Y_0^2 \tag{27}$$

Clearly, the matrix shown in Eq. (24) is constant when there are no period errors (i.e., Y_0 is zero). But, even with period errors, the "form" of the matrix remains the same although some individual components (those containing X_t) may grow numerically. In essence, Eq. (24) represents an analytical expression for the covariance matrix as a function of time and, indeed, it may be verified that this covariance matrix satisfies the standard propagation by transition matrix formula

$$E_t = \Phi E_0 \Phi^T \tag{28}$$

with E_0, the initial covariance ($t = 0$), given by

$$E_0 = \begin{bmatrix} X_0^2 + 2b^2 & kX_0Y_0 & 0 & -(3\omega/2)\,kX_0Y_0 & \omega b^2 & 0 \\ kX_0Y_0 & Y_0^2 + (1/2)b^2 & 0 & -\omega[(3/2)Y_0^2 + b^2] & 0 & 0 \\ 0 & 0 & (1/2)c^2 & 0 & 0 & 0 \\ -(3\omega/2)kX_0Y_0 & -\omega[(3/2)Y_0^2 + b^2] & 0 & \omega^2[(9/4)Y_0^2 + 2b^2] & 0 & 0 \\ \omega b^2 & 0 & 0 & 0 & (\omega^2/2)b^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\omega^2/2)c^2 \end{bmatrix} \tag{29}$$

The initial covariance matrix is especially important because it is used to initialize a simulation. The initialization would be accomplished by choosing values for b, c, and X_0 that may be thought of as the size of the error ellipsoid in position space (radial, crosstrack, and downrange, respectively), then choosing the period error through Y_0, and finally some value for k the correlation between X_0 and Y_0.

Most Common Special Case

Although it is permissible for the parameters X_0 and Y_0 to be correlated, it is difficult to imagine what that correlation might be without a priori knowledge. Thus, it seems perfectly reasonable to assume that no correlation is the most likely situation in practice. Furthermore, setting k to zero in Eq. (29) yields a particularly simple result:

$$E = \begin{bmatrix} X_0^2 + 2b^2 & 0 & 0 & 0 & \omega b^2 & 0 \\ 0 & Y_0^2 + (1/2)b^2 & 0 & -\omega[(3/2)Y_0^2 + b^2] & 0 & 0 \\ 0 & 0 & (1/2)c^2 & 0 & 0 & 0 \\ 0 & -\omega[(3/2)Y_0^2 + b^2] & 0 & \omega^2[(9/4)Y_0^2 + 2b^2] & 0 & 0 \\ \omega b^2 & 0 & 0 & 0 & (\omega^2/2)b^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\omega^2/2)c^2 \end{bmatrix} \tag{30}$$

Furthermore, its legitimacy as a covariance matrix may be readily established, a property that will then be conserved under propagation and/or measurement incorporation. The correlation matrix corresponding to this matrix may be written

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & c_{x\dot{y}} & 0 \\ 0 & 1 & 0 & c_{y\dot{x}} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & c_{y\dot{x}} & 0 & 1 & 0 & 0 \\ c_{x\dot{y}} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{31}$$

where the downrange position, vertical velocity is given by

$$c_{x\dot{y}} = \frac{b}{\sqrt{b^2 + \frac{1}{2}X_0^2}} \tag{32}$$

and the vertical position, downrange velocity is given by

$$c_{y\dot{x}} = \frac{-\left(b^2 + \frac{3}{2}Y_0^2\right)}{\sqrt{\left(b^2 + \frac{3}{2}Y_0^2\right)^2 + \frac{1}{8}b^2Y_0^2}} \tag{33}$$

Note that vertical position and downrange velocity are perfectly correlated when there is no period error (i.e., Y_0 is zero), and downrange position and vertical velocity are perfectly correlated when there is no downrange error (i.e., X_0 is zero). Typically, a pure downrange offest X_0 has no effect on orbital mechanics and, as a result, tends to be more difficult to observe. Of course, 100% correlations are not realistic, especially since circular orbits do not exist in practice. Clearly the correlations shown here should probably be tempered somewhat. This is easily achieved by assuming at least small values for X_0 and Y_0.

The form of Eq. (30) is especially appealing in that it really is comprised of three (2 × 2) submatrices and, as such, the corresponding characteristic equation is easily written as three separate quadratics:

$$\lambda^2 - [X_0^2 + (2 + \tfrac{1}{2}\omega^2)b^2]\lambda + \tfrac{1}{2}\omega^2 b^2 X_0^2 = 0 \tag{34}$$

$$\lambda^2 - [(1 + \tfrac{9}{4}\omega^2)Y_0^2 + (\tfrac{1}{2} + 2\omega^2)b^2]\lambda + \tfrac{1}{8}\omega^2 b^2 Y_0^2 = 0 \tag{35}$$

$$(\lambda - \tfrac{1}{2}c^2)(\lambda - \tfrac{1}{2}\omega^2 c^2) = 0 \tag{36}$$

Clearly, the determinant is only zero (i.e., $\lambda = 0$) when X_0 or Y_0 or c is zero. Furthermore, with these equations it can also be shown that the eigenvalues are never negative thus ensuring that the matrix is indeed positive semidefinite, a necessary condition for a realistic covariance matrix. As a matter of convenience, the results of this section are reformulated in a nonrotating local-vertical coordinate system in the Appendix.

Conclusions

It is possible to devise a covariance matrix that remains constant, or grows as it should if there is a period error, in a rotating local-vertical coordinate system. The solution presented here should prove useful for initializing simulation covariance matrices for near-circular orbit problems.

Appendix: Nonrotating Local-Vertical Formulation

The conversion between state deviations in rotating and nonrotating local-vertical coordinates is a simple matter of adding (or subtracting) a Coriolis term to the velocity component

$$\Delta \underline{r}_R = \Delta \underline{r}_{NR} \tag{A1}$$

$$\Delta \underline{v}_R = \Delta \underline{v}_{NR} - \underline{\omega} \times \Delta \underline{r}_{NR} \tag{A2}$$

and

$$\Delta \underline{r}_{NR} = \Delta \underline{r}_R \tag{A3}$$

$$\Delta \underline{v}_{NR} = \Delta \underline{v}_R + \underline{\omega} \times \Delta \underline{r}_R \tag{A4}$$

Here, the subscript R denotes rotating, and NR nonrotating. In a six-dimensional format, these equations become

$$\Delta \underline{x}_R = \begin{bmatrix} I & 0 \\ -\Omega & I \end{bmatrix} \Delta \underline{x}_{NR} \tag{A5}$$

$$\Delta \underline{x}_{NR} = \begin{bmatrix} I & 0 \\ \Omega & I \end{bmatrix} \Delta \underline{x}_R \tag{A6}$$

The angular velocity vector is given by

$$\underline{\omega} = -\omega \underline{u}_z = -\omega \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{A7}$$

so that the cross-product matrix Ω may be written

$$\Omega = [\underline{\omega} \times] = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A8}$$

with

$$\Omega^T = -\Omega \tag{A9}$$

From these expressions, it follows that the transition matrix and the covariance matrix may be transformed into a local-vertical, but nonrotating, format by means of

$$\Phi_{NR} = \begin{bmatrix} I & 0 \\ \Omega & I \end{bmatrix} \Phi_R \begin{bmatrix} I & 0 \\ -\Omega & I \end{bmatrix} \tag{A10}$$

$$E_{NR} = \begin{bmatrix} I & 0 \\ \Omega & I \end{bmatrix} E_R \begin{bmatrix} I & -\Omega \\ 0 & I \end{bmatrix} \tag{A11}$$

Multiplying these out [using the results from Eqs. (6) and (30) for the quantities Φ_R and E_R, respectively] yields the expressions

$$\Phi_{NR} = \begin{bmatrix} 2\cos\omega t - 1 & 2\sin\omega t - 3\omega t & 0 & (1/\omega)(4\sin\omega t - 3\omega t) & (2/\omega)(\cos\omega t - 1) & 0 \\ \sin\omega t & 2 - \cos\omega t & 0 & (2/\omega)(1-\cos\omega t) & (1/\omega)\sin\omega t & 0 \\ 0 & 0 & \cos\omega t & 0 & 0 & (1/\omega)\sin\omega t \\ -\omega\sin\omega t & \omega(\cos\omega t - 1) & 0 & 2\cos\omega t - 1 & -\sin\omega t & 0 \\ \omega(1-\cos\omega t) & \omega(3\omega t - \sin\omega t) & 0 & 3\omega t - 2\sin\omega t & 2 - \cos\omega t & 0 \\ 0 & 0 & -\omega\sin\omega t & 0 & 0 & \cos\omega t \end{bmatrix} \tag{A12}$$

and at the initial time with zero correlation coefficient k,

$$E_{NR} = \begin{bmatrix} X_0^2 + 2b^2 & 0 & 0 & 0 & -\omega(X_0^2 + b^2) & 0 \\ 0 & Y_0^2 + (1/2)b^2 & 0 & -(\omega/2)(Y_0^2 + b^2) & 0 & 0 \\ 0 & 0 & (1/2)c^2 & 0 & 0 & 0 \\ 0 & -(\omega/2)(Y_0^2 + b^2) & 0 & (\omega^2/4)(Y_0^2 + 2b^2) & 0 & 0 \\ -\omega(X_0^2 + b^2) & 0 & 0 & 0 & \omega^2[X_0^2 + (1/2)b^2] & 0 \\ 0 & 0 & 0 & 0 & 0 & (\omega^2/2)c^2 \end{bmatrix} \tag{A13}$$

The corresponding correlation matrix is given by

$$C_{NR} = \begin{bmatrix} 1 & 0 & 0 & 0 & c_{x\dot{y}} & 0 \\ 0 & 1 & 0 & c_{y\dot{x}} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & c_{y\dot{x}} & 0 & 1 & 0 & 0 \\ c_{x\dot{y}} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A14}$$

where the downrange position, vertical velocity is given by

$$c_{x\dot{y}} = \frac{-(b^2 + X_0^2)}{\sqrt{(b^2 + X_0^2)^2 + \tfrac{1}{2} b^2 X_0^2}} \tag{A15}$$

and the vertical position, downrange velocity is given by

$$c_{y\dot{x}} = \frac{-(b^2 + Y_0^2)}{\sqrt{(b^2 + Y_0^2)^2 + \tfrac{1}{2} b^2 Y_0^2}} \tag{A16}$$

The form of the above matrices is sometimes useful because they are not corrupted by Coriolis effects. However, beyond that, they tend to be notationally less compact, and not terribly illuminating.

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